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1986 J. Phys. A: Math. Gen. 19 2679

(http://iopscience.iop.org/0305-4470/19/13/033)

J. Phys. A: Math. Gen. 19 (1986) 2679-2684. Printed in Great Britain

COMMENT

Density of states of a Sierpinski gasket in two dimensions with anisotropic interactions

M Hood† and B W Southern‡§

† Department of Physics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2
‡ CEN Saclay, Service de Physique Théorique, 91191 Gif-sur-Yvette, Cedex, France

Received 2 December 1985, in final form 16 January 1986

Abstract. The density of states of a quadratic Hamiltonian, with some anisotropy in the interactions, on a Sierpinski gasket in two dimensions is calculated. It is shown that the isotropic gasket is special in that the spectrum is composed of two distinct parts which correspond to localised molecular modes and hierarchical modes respectively. When anisotropy is added, the localised molecular modes disappear and only the hierarchical states remain.

Recently there has been a growing interest in the spectral properties of self-similar or fractal lattices (Mandelbrot 1979), and they have been used as simple models of disordered systems. Because of their dilation symmetry many problems can be solved exactly on these lattices using renormalisation group methods (Dhar 1977). In particular the density of states (DOS) of a quadratic Hamiltonian, for example the tight-binding model, on the Sierpinski gasket in $d = 2$ has been studied by several groups (Domany *et al* 1983, Rammal 1983, 1984, Tremblay and Southern 1983, Southern and Douchant 1985). The DOS is a superposition of two distinct contributions: the first part consists of a discrete spectrum of localised molecular modes which have non-zero amplitudes on a finite number of sites whereas the second part is a Cantor set corresponding to modes with a hierarchy of localisation lengths (see Rammal (1984) for a detailed discussion).

In this comment we study the effects on the DOS of adding some anisotropy to the parameters of the system. The construction of the Sierpinski gasket is shown in figure 1. At each level three sites are inserted into each upward-pointing triangle and joined

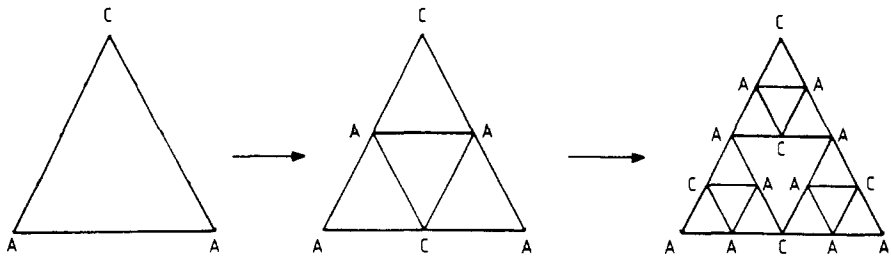

Figure 1. The construction of the Sierpinski gasket is shown, with the labelling of sites (A and C) appropriate to this problem.

§ Permanent address: Department of Physics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2.

0305-4470/86/132679+06$02.50

to obtain the next level. The labelling of sites refers to the problem considered here. The Green functions for the system are given by the equations

$$(E+i\eta-\varepsilon_i)G_{ij}=\delta_{ij}+\sum_l V_{il}G_{lj} \tag{1}$$

where $E+i\eta$ is the complex energy. We have $\varepsilon_i=\varepsilon_A$ if i is an A site, $\varepsilon_i=\varepsilon_C$ if i is a C site, $V_{il}=V_1$ if i and l are nearest-neighbour A sites, $V_{il}=V_2$ if i and l are nearest-neighbour A and C sites and $V_{il}=0$ otherwise. For the specific case of a ferromagnet with exchange interactions given by $-V_1$ and $-V_2$, we have

$$\begin{aligned} E_A &\equiv E+i\eta-\varepsilon_A=E+i\eta+2V_1+2V_2 \\ E_C &\equiv E+i\eta-\varepsilon_C=E+i\eta+4V_2. \end{aligned} \tag{2}$$

We use the formalism described by Tremblay and Southern (1983) and only present a brief outline and the necessary equations here. The generating function for this problem is

$$F=\ln\left(\int Du\,\exp\{(i/2)U^T[(E+i\eta)I-H]U\}\right) \tag{3}$$

where $U^T=(u_1, u_2, \ldots u_N)$, with u_i being a continuous variable at sites i on the lattice. In our case the $N\times N$ matrix H is given by

$$H_{ij}=\varepsilon_i\delta_{ij}+V_{ij}. \tag{4}$$

We calculate F using an exact rescaling procedure that involves a partial integration over the variables associated with the smallest triangles. F can then be written as

$$F=\ln\left(\int Du'\,\exp\{(i/2)U'^T[(E+i\eta)I-H'_{11}]U'\}\right)+NC' \tag{5}$$

with the integration over uneliminated sites only and

$$H'_{11}=H_{11}+H_{12}[(E+i\eta)I-H_{22}]^{-1}H_{21} \tag{6}$$

$$NC'=-\tfrac{1}{2}\ln\{\det[(E+i\eta)I-H_{22}]\}. \tag{7}$$

The uneliminated sites form a gasket in which all distances have been scaled by a factor of 2 and (6) yields recursion relations for V_1, V_2, E_A and E_C which can be iterated. If we define the dimensionless parameters

$$X_{1A}=V_1/E_A \qquad X_{2A}=V_2/E_A \qquad X_{2C}=V_2/E_C \tag{8}$$

then the new renormalised parameters are given by

$$\begin{aligned} X'_{1A} &= (X_{2A}X_{2C}+X_{1A}Y)/(1-3Y+2X_{2A}X_{2C}) \\ X'_{2A} &= (X_{1A}+Y)X_{2A}/(1-3Y+2X_{2A}X_{2C}) \\ X'_{2C} &= (X_{1A}+Y)X_{2C}/(1-3Y+2X_{1A}^2) \\ Y &= X_{1A}^2+2X_{2A}X_{2C}+2X_{1A}X_{2A}X_{2C}. \end{aligned} \tag{9}$$

The constant term in F above after l iterations is given by

$$C^{(l)}=C^{(l-1)}+[\ln(1-Y)-\ln(1-3Y+2X_{1A}^2)-2\ln(1-3Y+2X_{2A}X_{2C})]^{(l-1)}/(2\times 3^{l+1}) \tag{10}$$

with $C^{(0)}=-(2\ln E_A+\ln E_C)/6$.

The DOS is obtained by iterating the above recursion relations for F and then using the identity

$$\rho(E,\eta)=(2/\pi N)\,\mathrm{Im}[\partial F(E+i\eta)/\partial E]. \tag{11}$$

It is straightforward but tedious to differentiate both sides of (10) with respect to E using the chain rule. For any given value of η, V_1 and V_2 (and hence X_{1A}, X_{2A} and X_{2C}) iterate to zero and $\partial C^{(l)}/\partial E$ converges. We iterate $\partial C^{(l)}/\partial E$ numerically until it converges to the required accuracy and take the imaginary part to obtain the DOS.

Before discussing the results we make a few comments about special values of V_2/V_1. The isotropic case mentioned earlier is given by $V_1 = V_2$ and this spectrum has been discussed in detail by Rammal (1983, 1984). At low E, the DOS behaves as $E^{(\tilde{d}-2)/2}$ where $\tilde{d}$ is the spectral dimension. It is related to the fractal dimension D and the dynamic scaling exponent z by $\tilde{d} = 2D/z$. The state with the highest energy is a localised molecular mode located at $E = -6V_1$. In the opposite extreme of $V_2 = 0$, the system reduces to a chain of A sites with isolated C sites. The DOS for this case is continuous and is the superposition of the DOS for a linear chain, with square root singularities at the band edges, accounting for $\frac{2}{3}$ of the integrated DOS, and a δ function at $E = 0$ due to the free C sites with a weight of $\frac{1}{3}$.

One might expect the spectrum to evolve continuously from the isotropic limit to the chain limit as V_2/V_1 decreases. However, this is not the case as can be seen in figures 2-5 where the DOS is plotted as a function of E for various values of V_2/V_1 and $\eta = 0.01$. The localised molecular modes of the isotropic gasket disappear when

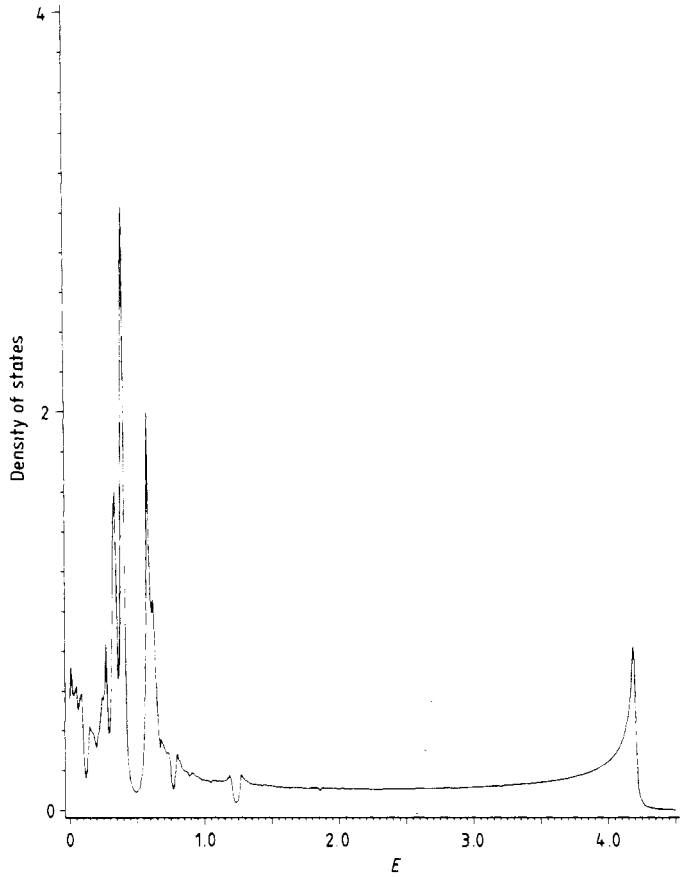

Figure 2. The DOS is shown as a function of E for $V_1 = -1$ and $V_2 = -0.1$. The imaginary part in the energy is $\eta = 0.01$.

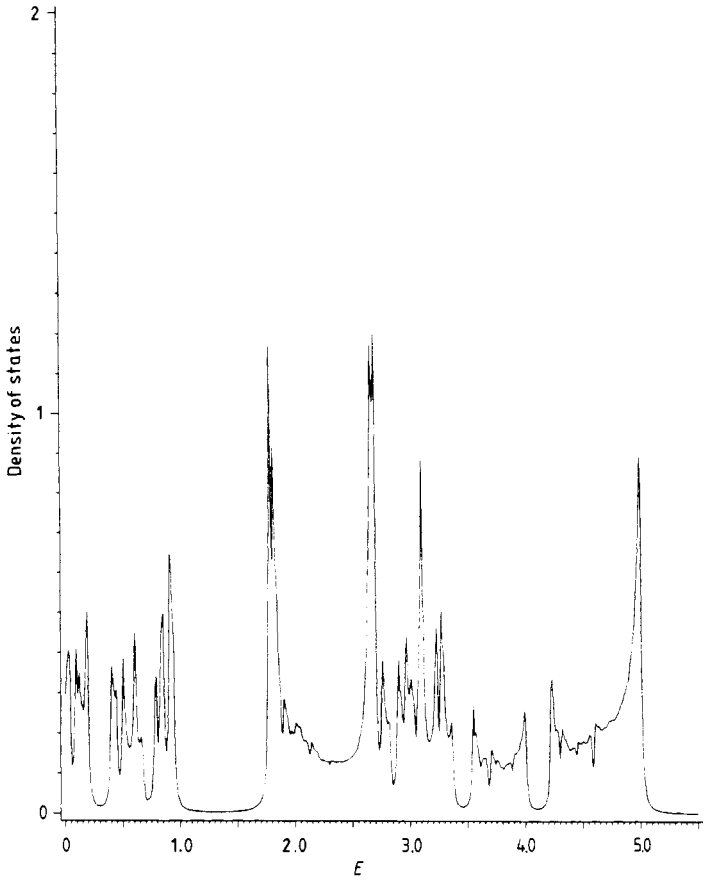

Figure 3. The DOS for $V_1 = -1$ and $V_2 = -0.5$ with $\eta = 0.01$.

V_2 is no longer equal to V_1. We also see that gaps and peaks appear in the spectra as soon as V_2 is non-zero.

Various features of the spectra can be understood in terms of the fixed points of the transformation in (9). For convenience we define a row vector $\boldsymbol{X} = (X_{1A}, X_{2A}, X_{2C})$. There is a trivial stable fixed point at $\boldsymbol{X}^* = (0, 0, 0)$ which is an attractive sink for all complex energies. There is an unstable fixed point at $\boldsymbol{X}^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, which is a fixed point of the isotropic case, with eigenvalues 5, 1 and $\frac{1}{5}$. The dynamic critical exponent z is related to the largest eigenvalue λ by $z = \ln \lambda / \ln 2$ whereas the fractal dimension D of the gasket is $D = \ln 3 / \ln 2$. This fixed point controls the behaviour of the DOS near $E = 0$ for all V_2 except the chain limit $V_2 = 0$. Hence the spectral dimension of the anisotropic gasket remains unchanged. There is a line of fixed points at $\boldsymbol{X}^* = (\frac{1}{2}, 0, X_{2C})$ with arbitrary X_{2C}, with eigenvalues 9, 3 and 1. The low E DOS is controlled by the fixed point located at $X_{2C} = 0$ on this line in the chain limit and has a spectral dimension of unity. A fixed point at $\boldsymbol{X}^* = (-\frac{1}{2}, 0, 0)$ with eigenvalues 9, -1, $-\frac{1}{3}$ describes the upper band edge of the chain which also has a square root divergence.

In addition to these fixed points, there is also a limit cycle (a fixed point with eigenvalues 81, 1 and $\frac{1}{9}$ under two iterations, at $\boldsymbol{X}^* = (-\frac{1}{2}, X_{2A}, 0)$ with arbitrary X_{2A}) which controls the behaviour of the DOS at the upper edge of the spectrum in the anisotropic case. The upper edge of the spectrum is at $E = -4V_1 - 2V_2$, which has

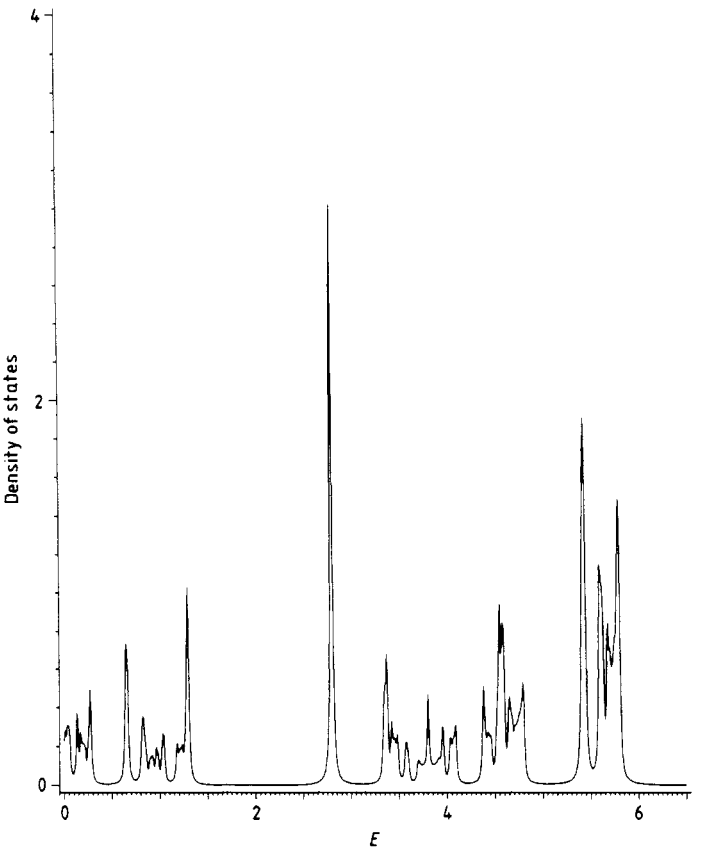

Figure 4. The DOS for $V_1 = -1$ and $V_2 = -0.9$ with $\eta = 0.01$.

$X_{1A} = -\frac{1}{2}$, and iterates towards the limit cycle mentioned above, except for the cases where $V_2 = V_1$ or $V_2 = -2V_1$, and the DOS has a square root divergence, as in the chain. The isotropic case $V_2 = V_1$ is not attracted to this limit cycle but has a local mode at $E = -6V_1$ whose weight in the DOS is $\frac{1}{3}$.

For the isotropic gasket, the localised molecular modes correspond to energies for which $V_2^{(l)} = E_C^{(l)} = 0$ at the lth step. However, for the anisotropic gasket, these conditions cannot be satisfied simultaneously. For example, the conditions $V_2' = E_C' = 0$ give two solutions; one is $V_2 = V_1$ and $E = -6V_1$, and the other is $V_2 = -2V_1$ and $E = 0$. However, the latter solution lies outside the range of values of the parameters for which the ferromagnetic ground state is stable.

For general V_2/V_1 the spectrum consists of bands and gaps. There is always a gap edge at $E = -3V_1 - 2V_2$ for which the recursion relations iterate to the fixed line $\boldsymbol{X}^* = (\frac{1}{2}, 0, X_{2C})$ as soon as V_2 is non-zero. Since the leading eigenvalue is 9, the DOS has a square root divergence at this edge. In the isotropic case this energy iterates to the isotropic fixed point which controls the $E = 0$ modes and the spectral dimension is $\ln 9/\ln 5$.

Hence we reach the unexpected conclusion that there are hierarchical modes present for the non-isotropic gasket but the molecular modes have disappeared. These modes are replaced by bands which have singularities at the edges of the inverse square root type. In addition, the upper band edges of the hierarchical modes have a different

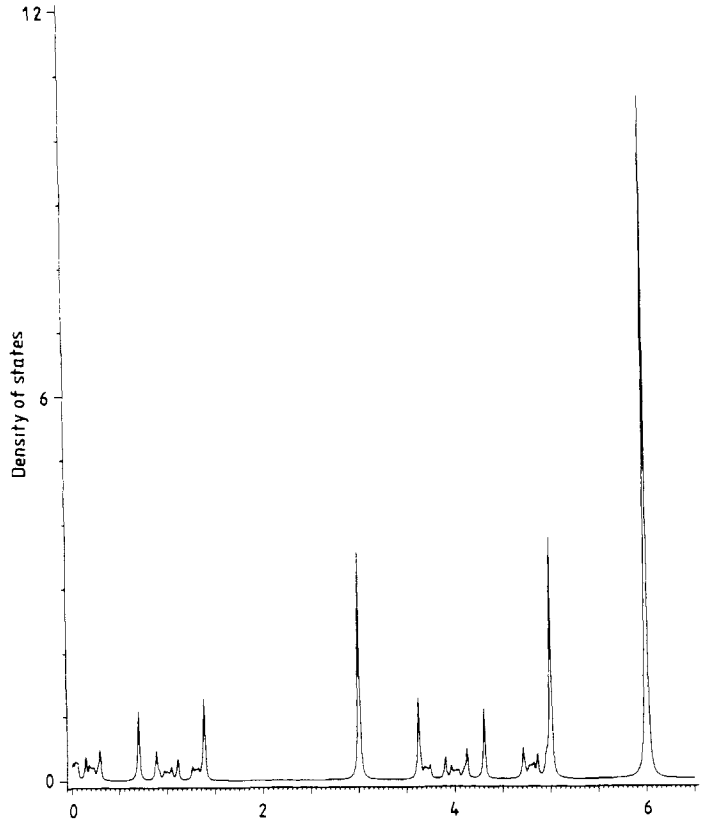

Figure 5. The DOS for $V_1 = V_2 = -1$, i.e. the isotropic limit, with $\eta = 0.01$.

behaviour in the anisotropic case. It appears that the isotropic gasket is in fact a very special case of the more general problem.

Acknowledgments

This work was funded by the Natural Sciences and Engineering Research Council of Canada. One of us (BWS) would like to thank the members of the Service de Physique Théorique, CEN Saclay, for their warm hospitality.

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